Algebra Problem

Notations

$\mathrm{Mat}_n(\mathbb{C})$ denotes the set of $n \times n$ matrices with complex coefficients,
$\mathrm{D}_n(\mathbb{C}) \subset \mathrm{Mat}_n(\mathbb{C})$ denotes the subset of diagonal matrices,
$\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{Mat}_n(\mathbb{C})$ denotes the group of invertible matrices,
$\mathrm{SL}_n(\mathbb{C})$ denotes the subgroup of $GL_n(\mathbb{C})$ of matrices of determinant 1,
I_n denotes the identity matrix of $\mathrm{Mat}_n(\mathbb{C})$,
$|X|$ denotes the cardinal of a set X.

1. Finite abelian subgroups of $SL_n(\mathbb{C})$

1. Let $x \in \mathrm{Mat}_n(\mathbb{C})$ such that $x^N = \mathrm{I}_n$ for some positive integer N. Show that x is diagonalizable.

2. Let $X \subset \mathrm{Mat}_n(\mathbb{C})$ be a subset of pairwise commuting diagonalizable matrices. Show that there exists $a \in \mathrm{GL}_n(\mathbb{C})$ such that $aXa^{-1} \subset \mathrm{D}_n(\mathbb{C})$.

3. Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C}) \cap \mathrm{D}_n(\mathbb{C})$. We denote by $a_{i,j}$ the coefficients of a matrix a. We define a set of complex numbers:

$$\Lambda = \{g_{1,1} \ : \ g \in G\} \ \subset \mathbb{C} \ .$$

   (a) Show that the finite subgroups of the multiplicative group $(\mathbb{C} \setminus \{0\}, \cdot)$ are cyclic. Deduce that Λ is a cyclic group.

   (b) Let λ be a generator of Λ and let $a \in G$ such that $a_{1,1} = \lambda$. We denote by G_1 the subgroup of G generated by a. We also denote by G' the subgroup of G of matrices g with $g_{1,1} = 1$.
   Show that the group G is isomorphic to the product $G_1 \times G'$.

4. Let G be a finite abelian subgroup of $\mathrm{SL}_n(\mathbb{C})$. Show that there is an integer $k \leq n-1$ and k cyclic groups C_i, such that G is isomorphic to the product $C_1 \times \cdots \times C_k$.

2. Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$

We identify the endomorphisms of $\mathbb{C}^2$ with their matrices in the canonical basis of $\mathbb{C}^2$. We denote by $\mathbb{P}^1$ the set of dimension one vector subspaces of $\mathbb{C}^2$:

$$\mathbb{P}^1 = \{\ell \subset \mathbb{C}^2 \ : \ \dim \ell = 1\} \ .$$

If $a \in \mathrm{GL}_2(\mathbb{C})$, we denote by ψ_a the map:

$$\begin{array}{rccl} \psi_a : & \mathbb{P}^1 & \to & \mathbb{P}^1 \\ & \ell & \mapsto & a(\ell) = \{a(v) \ : \ v \in \ell\} \end{array}$$

Finally we denote by $\mathrm{Aut}(\mathbb{P}^1)$ the set of all maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ which are of the form $f = \psi_a$ for some $a \in \mathrm{GL}_2(\mathbb{C})$.

1. Show that $\mathrm{Aut}(\mathbb{P}^1)$ is a group (for the composition of applications). Show that the application $\begin{array}{rccc} \psi : & \mathrm{SL}_2(\mathbb{C}) & \to & \mathrm{Aut}(\mathbb{P}^1) \\ & a & \mapsto & \psi_a \end{array}$ is a surjective morphism of groups and compute its kernel.

2. Let $f \in \mathrm{Aut}(\mathbb{P}^1)$ be an element of order $n \geq 2$. Show that f has exactly 2 fixed points.

3. (**Finite subgroups of** $\mathrm{Aut}(\mathbb{P}^1)$) Let G be a finite subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ with at least two elements. We define a set:

$$X = \{\ell \in \mathbb{P}^1 \ : \ \exists g \in G \setminus \{\mathrm{Id}\} \quad g(\ell) = \ell\} \ .$$

   (a) Show that X is a finite nonempty set, and that if $g \in G$ and $\ell \in X$, then $g(\ell) \in X$.

   So the group G acts on X by $g \cdot \ell := g(\ell)$. We denote by $\mathcal{O}_1, \ldots, \mathcal{O}_N$ the orbits of this action, and we let $n_i = |G|/|\mathcal{O}_i|$.

   (b) Let $E = \{(g, \ell) \in G \times X \ : \ g \cdot \ell = \ell\}$. Show the equality:

$$|X| + 2(|G| - 1) = |E| = N|G| \ . \qquad (E_1)$$

   Deduce the equality:

$$\sum_{i=1}^{N} \left(1 - \frac{1}{n_i}\right) = 2 - \frac{2}{|G|} \ . \qquad (E_2)$$

   (c) Show that N and the n_i's are integers greater or equal to 2. Deduce that $N = 2$ or 3.

   (d) Assume that $N = 2$. Show that G is cyclic.

   (e) Assume that $N = 3$. Show that one of the n_i's is equal to 2, and deduce that $|G|$ is even.

4. (**Finite subgroups of** $\mathrm{SL}_2(\mathbb{C})$) Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Assume that G is not cyclic. Show that $|G|$ is a multiple of 4. *Hint: show first that a finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ is cyclic if and only if $\psi(G)$ is cyclic.*