Physics

Exercise 1: Optical properties of pigments

The aim of this problem is to discuss the optical properties of pigments in simple macroscopic terms. A pigment is a chemical substance that is not soluble in the medium that it colors. In order to obtain a paint, pigments – in the form of powder – are put in suspension in a binder. From the point of view of the propagation of the light, a pigment is a diffuse scatterer. Such a medium corresponds to a binder in which a myriad of objects of a size non-negligeable compared to the wave length of the light are dispersed. A single object can either scatter the light, or absorb it. The reflectivity of a layer of pigment of thickness L placed on a substrate (of reflectivity R_S) is a problem that is very difficult to solve in a microscopic fashion since the size of the objects that make up the powder gives it a sizeable diffusion coefficient. In the following, we will propose a one-dimensional modelisation (cf figure 1). The light intensity is given by two fluxes J_- and J_+ which propagate respectively towards the substrate or in the opposite direction. We assume that the absorption of light by the pigment particles is described by an absorption coefficient K and that the scattering is given by a coefficient β.

Part I.

We give the general form of the equations gouverning this model for $0 \leq x \leq L$ (beware, however, that the indices of the different fluxes have been replaced by ? and will have to be retrieved!):

$$\begin{aligned} \frac{dJ_?}{dx} &= -(K+\beta)J_? + \beta J_? \\ -\frac{dJ_?}{dx} &= -(K+\beta)J_? + \beta J_? \end{aligned} \tag{1}$$

(1) Write the correct equations.
(2) Explain the physical meaning of the different terms.
(3) Give the units of K and β.
(4) Derive a differential equation for the ratio $R = J_+/J_-$.

Show that this equation can be cast into the form

$$\frac{dR}{dx} = P(R) \tag{2}$$

where $P(R)$ is a polynomial of second order.
(5) What is the physical meaning of R? What is its unit?
(6) Which value should $R(x)$ take on for $x = 0$?

Part II

The aim of this part is to find R at the surface of the layer, that is for $x = L$. We will denote this quantity as R_L.
(7) Write an integral equation that contains R_L and $R(x = 0)$ as boundaries. NB. Hint: separation of variables!
(8) Integrate this equation and show that R_L can be written as

$$R_L = \frac{R_\infty^{-1} - R_\infty \frac{R_S - R_\infty^{-1}}{R_S - R_\infty} e^{\beta L (R_\infty^{-1} - R_\infty)}}{1 - \frac{R_S - R_\infty^{-1}}{R_S - R_\infty} e^{\beta L (R_\infty^{-1} - R_\infty)}}. \tag{3}$$

with $\alpha = K/\beta + 1$ and $R_\infty = \alpha - \sqrt{\alpha^2 - 1}$. (What is R_∞^{-1} ?)

Part III

(9) Discuss the physical meaning of R_∞.
(10) Sketch R_∞ as a function of α and interpret it physically.
(11) Calculate R_L for the following two cases:
(a) $R_S = 0$
(b) $R_S = 1$.
(12) To which physical situations do the cases (11a) and (11b) correspond?
(13) In the cases (11a) and (11b), which of the two quantities R_L and R_∞ is the larger one?

Part IV

In reality, the absorbing power of a pigment depends on the frequency of the light, and thus $K = K(\omega)$. We assume that K behaves as $K(\omega) = k\sqrt{\omega - \omega_0}$ for $\omega > \omega_0$ and that it vanishes for $\omega < \omega_0$.
(14) Which physical situation is described by such a dependence?

(15) An example of a red pigment is mercury sulphide, HgS. In your opinion, what is the order of magnitude of ω_0 ?

(16) A measurement of R_∞ as a function of frequency gives – for $\omega > \omega_1$ – the following dependence

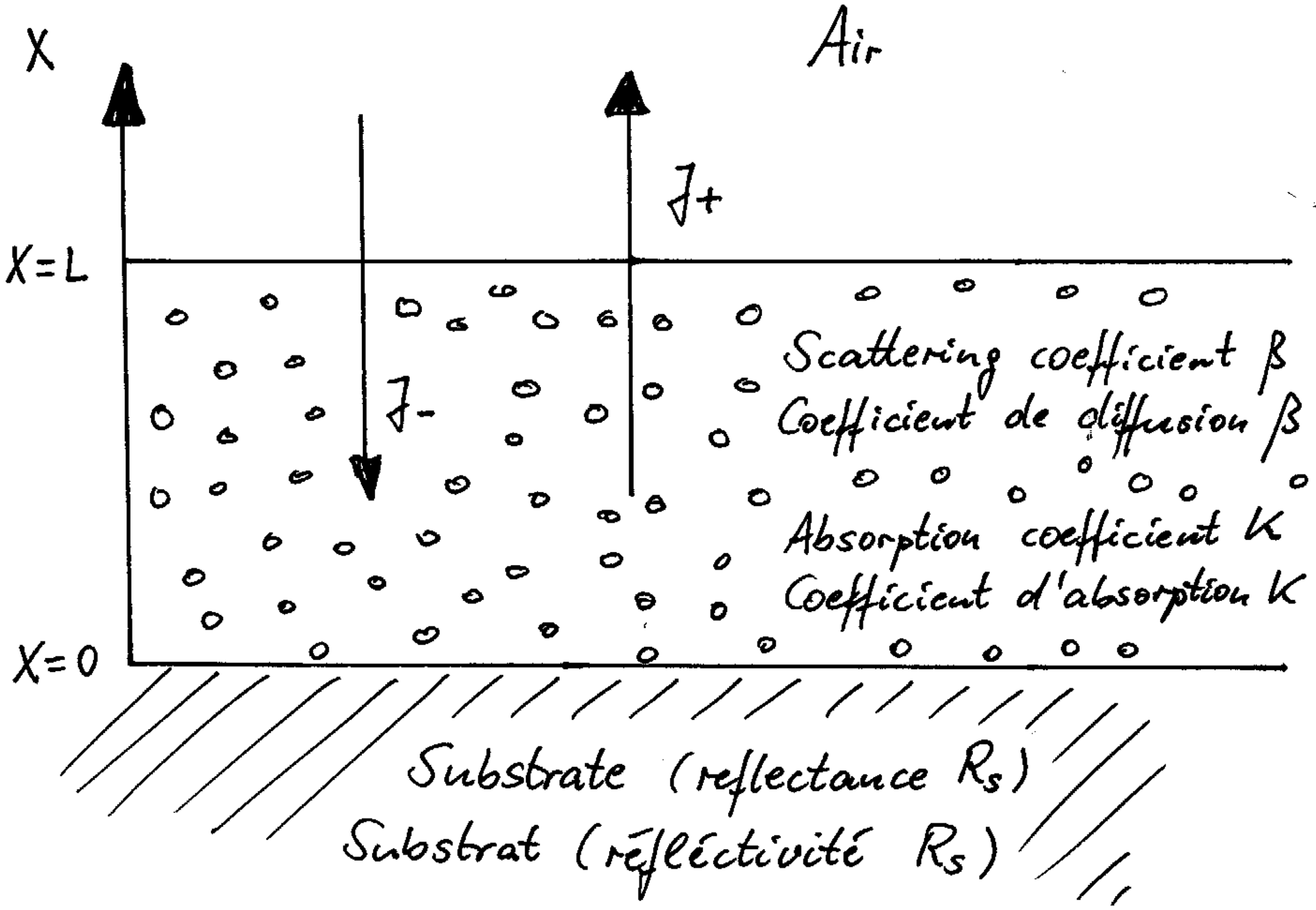
$$R_\infty(\omega) = 1 + C\sqrt{\omega - \omega_1} - \sqrt{(1 + C\sqrt{\omega - \omega_1})^2 - 1} \tag{4}$$

with constants C and ω_1. What can you conclude about β and ω_0?

(17) What would you expect for $R_\infty(\omega)$ for $\omega < \omega_1$? Why?

Part V

(18) Is there a fundamental notion of wave optics that is absent from the description that we have worked out in this exercise? Which one?


X
Air
J+
X=L
Scattering coefficient β
Coefficient de diffusion β
J-
Absorption coefficient K
Coefficient d'absorption K
X=0
Substrate (reflectance Rs)
Substrat (réfléctivité Rs)

Exercise 2: Classical Mechanics

Part I. Rigid body dynamics

Consider a solid cylinder of length L, radius R, and mass M rolling down an inclined plane without slipping. The inclination angle of the plane is θ. The axis of the cylinder is assumed to remain horizontal throughout the motion.

1. Compute the moment of inertia of the cylinder around its central axis. Recall that for a collection of particles of mass m_i at distance r_i from the rotation axis, the moment of inertia is defined as $\sum_i m_i r_i^2$. For solid bodies, you must take the continuum limit.
2. What are the translational and rotational kinetic energy of the cylinder? Express both in terms of M and the velocity of the center of mass of the cylinder.
3. What is its potential energy?
4. The cylinder is released at the top of the plane at height h_0 with no initial velocity. By using energy conservation, determine its terminal velocity as it reaches the bottom of the inclined plane.
5. By analyzing the forces acting on the cylinder, determine the position of its center of mass as a function of time, $x(t)$. Use this result to recompute the terminal velocity of the cylinder.

The equations of motion governing the trajectory $x(t)$ of the center of mass of the cylinder can also be determined by using the Lagrangian formalism. The Lagrangian L of a system is defined as the difference between its kinetic and its potential energy, $L = T - V$. It is considered as a function of pairs of variables: one variable for each degree of freedom, together with its time derivative. For the system at hand, this translates into L being a function of $x(t)$ and $\dot{x}(t)$. The equations of motion are then given by the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0\,. \tag{0.1}$$

For the purpose of taking the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial \dot{x}}$, x and $\dot{x}$ are considered as independent variables.

6. Compute the Lagrangian of the system.
7. Determine and solve the corresponding Euler-Lagrange equation.

Part II. A pulley system

Consider the pulley system depicted in figure 1. We will ignore friction as well as all masses

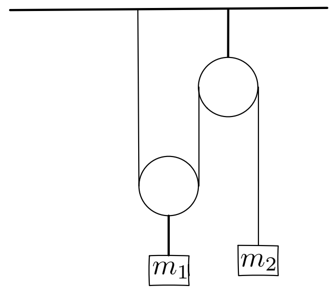


Figure 1: A pulley system.

in the problem except for those indicated, and assume that the rope threading the pulleys does not stretch.

1. Determine the trajectory $x(t)$ of the block of mass m_1 by studying the forces acting on the two blocks.
2. Redo the calculation in the Lagrangian formalism (as introduced in the previous problem).

Part III: Charged beads on rods

Two beads of mass m, M and charge q, Q respectively can move without friction along two rods a distance d apart, as depicted in figure 2.

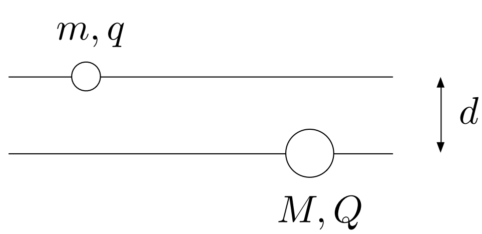


Figure 2: Two charged beads.

1. By studying the forces acting on the two beads, determine the system of differential equations governing their motion.
2. Analyze the system in the Lagrangian formalism. Note that the Lagrangian of a system with several degrees of freedom depends on multiple variables and the corresponding time derivatives, one pair per degree of freedom. Each degree of freedom has its own Euler-Lagrange equation, $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$.
3. What information about the motion of the system can you extract from the equations of motion?

Part IV: Beads connected by a spring

Two beads of equal mass are connected by a spring of spring constant k. They move without friction along two fixed rods as depicted in figure 3. We will assume in the following that the spring remains horizontal throughout the motion of the system.

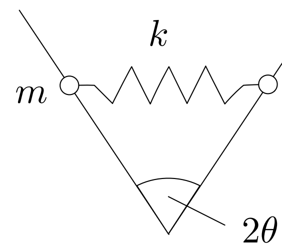


Figure 3: Two beads connected by a spring.

1. Neglecting gravity, determine the frequency of oscillation of the motion of the two beads by studying the forces acting on them.
2. Rederive the equations of motion in the Lagrangian formalism.
3. Now determine the motion of the beads in the presence of gravity. What aspect of the motion changes qualitatively?

Part V: Block on moving inclined plane

A block of mass m slides without friction on an inclined plane of mass M and inclination angle θ. The plane resides on a frictionless horizontal surface. The setup is depicted in figure 4. What is the acceleration of the plane?

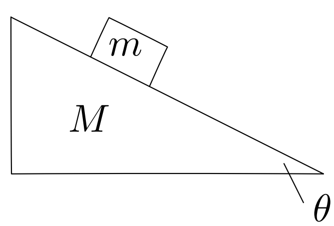


Figure 4: Block on inclined plane.

Part VI: Two masses connected by a string

A mass m is free to move on a frictionless surface. It is connected via a string to a mass M hanging below, as depicted in figure 5. Assume that the mass M can only move vertically, and that the string remains taut throughout the motion.

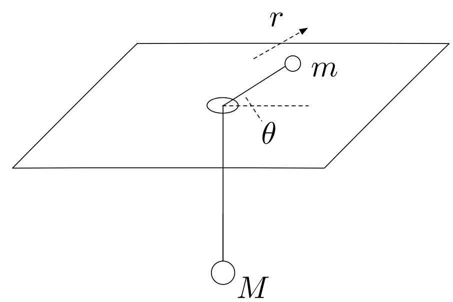


Figure 5: Two masses connected by a string.

1. Find the equations of motion for the variables r and θ depicted in the figure. Interpret the two equations physically.
2. Under what conditions does m undergo circular motion? Express the radius of the motion in terms of a conserved quantity.
3. What is the frequency of small oscillations (in the variable r) about this circular motion?