Entrance examination ENS SIS 2013
Physics
Michelson interferometer

This problem is composed of three parts : a general introduction about interferences created by two coherent point sources in a peculiar configuration (part 1), the use of a Michelson interferometer for spectroscopy (part 2) and the conception of a wavelength meter (part 3).

Generally, a light point source located in S emits, in a pulsed manner, light wavetrains which are supposed to have the same angular velocity ω. In scalar representation, monochromatic light wave is then characterized, at point M and time t, by the electric field $E(M,t) = a\cos\left[\omega t - \phi(M,t)\right]$, where a is the constant field amplitude and $\phi(M,t)$ the dephasing, at point M and time t, with respect to the reference point S. The complex quantity associated with $E(M,t)$ is : $\underline{E}(M,t) = a\exp\left[i\left(\omega t - \phi(M,t)\right)\right]$, where $i^2 = -1$.

The wavetrain model assumes that the phase at the point source ϕ_S remains constant during time intervals of constant duration τ_C between which the phase value changes randomly. The wave emitted during this time interval called the coherence time is a "wavetrain". The wavetrain is thus limited in time and propagates in vacuum at speed $c = 3\times10^8$ m.s^{-1}. The coherence time is the average travelling duration of the wavetrains at a given point. The wave phase at the source ϕ_S takes a new random value at each new wavetrain.

In the following, the light intensity $I(M)$ is measured by a detector located at M which is sensitive to the time average value of $E^2(M,t)$. The intensity is conventionally defined (except for a multiplicative constant) by :

$$I(M) = \langle \underline{E}(M,t)\underline{E}^*(M,t)\rangle = \langle |\underline{E}(M,t)|^2\rangle.$$

1 Preamble : interference pattern created by two coherent monochromatic point sources

Two coherent point sources S_1 and S_2 emit in vacuum two monochromatic waves of same wavelength λ_0, same amplitude a_0 and in phase at their respective origins S_1 and S_2. These sources, apart from a distance b, are symmetrical with respect to C (see Figure 1).

An observation screen (E) is placed perpendicularly to the line connecting S_1 and S_2 at a distance D from the point C between the two sources. The perpendicular line to the screen passing through C defines the system axis and this axis intersects the screen in B. The sources are in the plane (BXZ). The observation point M in the coordinate system $(BXYZ)$ is defined by $\rho = BM$, with $D \gg b$ and $D \gg \rho$.
The intensity observed on the screen hiding one of the two sources is denoted $I_0 = a_0^2$.

1. How do we realize such sources ?
2. Give the intensity $I(M)$ at point M as a function of the path difference $\delta(M) = (S_1M) - (S_2M)$, λ_0 and I_0.
3. Express $\delta(M)$ as a function of b and the angle $\theta = (\overrightarrow{CB}, \overrightarrow{CM})$.

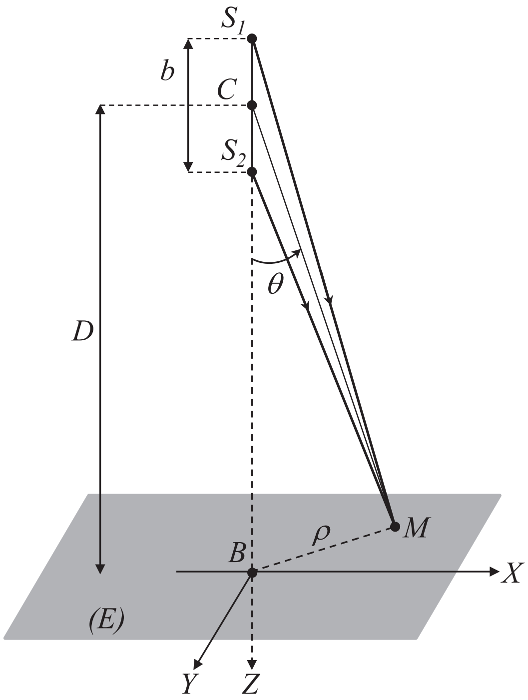

FIGURE 1 –

4. Deduce the intensity $I(M)$ at point M as a function of θ, b, λ_0 and I_0 ; then as a function of ρ, D, b, λ_0 and I_0.
5. What is the shape of the observed interference fringes ?
6. Define the interference order $p(M)$ at point M. Is it increasing or decreasing from the center B ?

2 The Michelson interferometer : a tool for spectroscopy

Figure 2 schematizes the Michelson interferometer principle. The mirrors are aligned so that circular fringes are observed on screen (E) which is located in the focal plane of a convergent lens (L) ("thin film" configuration of the Michelson interferometer). The lens focal distance is denoted f' and its axis Oz passes trough the screen at B ($OB = D$).

The interferometer is composed of :
- a beam splitter (S_P) which reflects half of the light ; the origin O of the coordinate system $(Oxyz)$ is centered on the beam splitter which makes a fixed angle of $\pi/4$ with the axis Ox and Oz. The dephasing introduced by the beam splitter is compensated by a compensating plate (not shown in the figure) paralleled to the beam splitter.
- two plane mirrors (M_1) and (M_2), perpendicular to the figure plane, of fixed orientations with respect to the beam splitter (S_P) with a corresponding angle $\pi/4$. Mirror (M_1) can be translated in the Oz direction whereas mirror (M_2) remains fixed. The distance between mirror (M_2) and origin O is denoted L_0.

Only waves which have been reflected once and only once on the beam splitter will be considered. Starting from the configuration where the image of (M_2) with respect to the beam splitter (S_P) coincides with (M_1), mirror (M_1) is translated by length e which is counted positively if the mirror moves away from the beam splitter.

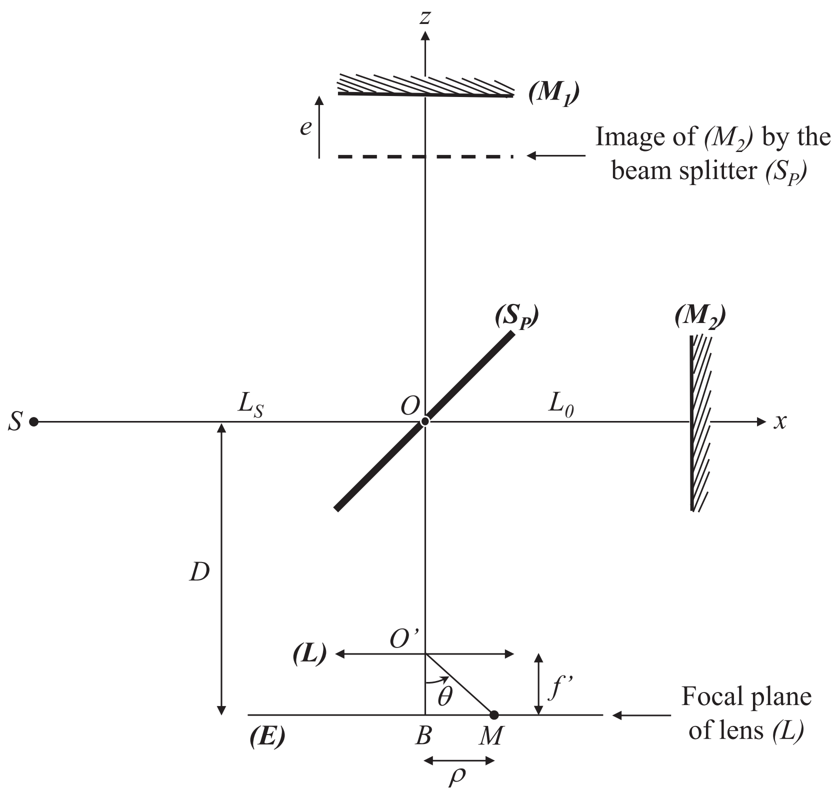

FIGURE 2 – Scheme of a Michelson interferometer in the "thin film" configuration.

2.1 Circular fringes of equal inclination

The monochromatic point source S of wavelength λ_0 is placed at a finite distance $L_S = SO$ from the beam splitter. The optical system composed of (S_P), (M_1) and (M_2) gives two images S_1 and S_2 of source S : S_1 corresponds to the beams hitting (M_1) and S_2 to the beams hitting (M_2). The intensity observed on the screen hiding one of the two sources is denoted I_0.

1. By symmetry arguments, place on a scheme the secondary sources S_1 and S_2, images of source S, and specify their coordinates in the coordinate reference (Oxz). Deduce the expression of $\overrightarrow{S_1S_2}$ as a function of e.
2. The distance between points M and B on screen (E) is denoted $\rho = BM$. Taking into account the condition $\rho \ll f'$, express the path difference $\delta(M)$ (defined here as a positive parameter) as a function of e and angle $\theta = (\overrightarrow{O'B}, \overrightarrow{O'M})$; then as a function of ρ, e, and f'. Establish, as a function of e, the path difference $\Delta = \delta(B)$ at B for $\rho = 0$.
3. Express the intensity $I(M)$ at point M as a function of ρ, e, f', λ_0 and I_0. Deduce the interference pattern projected on (E).
4. The center B of the circular fringes corresponds to a maximum of intensity. What is the interference order p_0, supposed to be an integer, at the center of the circular fringes ? Give the expression of the radius ρ_k of the k^{st} bright circular fringe counted from the center as a function of f', p_0 and its interference order p_k.
5. Express ρ_k as a function of k and ρ_1, radius of the first circular fringe counted from the center. Comment the interference pattern.
6. Describe (and justify) the fringes evolution when the thin film thickness value e is progressively increased. Do the circular fringes seem to "enter" or "leave" the center ? Is there an increasing or decreasing number of circular fringes visible on the screen ?

2.2 Interferograms : case of a monochromatic light

Mirror (M_1) can move from $e = 0$ to $e = L_{\max}$ ($L_{\max} > 0$). A point detector is placed at center B of the interference pattern. This detector gives an electrical signal proportional to the detected intensity and dependent on the path difference Δ. Increasing e induces a variation of the optical path at B from $\Delta = 0$ to $\Delta = \Delta_{\max}$ and, consequently, a scrolling of the circular fringes. One calls interferogram $I(\Delta)$ the recording of the intensity evolution I as a function of Δ.

The interferometer is illuminated by a monochromatic point source of wavelength λ_0. Express the intensity $I(\Delta)$ and show on a graph the interferogram $I(\Delta)$ as a function of Δ with its characteristic parameters.

2.3 Interferograms : case of a non-monochromatic light

2.3.1 Sodium doublet

We put at S a sodium lamp emitting with same intensity two monochromatic waves of wavelengths λ_1 and λ_2 close to the average wavelength $\lambda_0 = \frac{1}{2}(\lambda_1 + \lambda_2)$, with $\delta\lambda = (\lambda_2 - \lambda_1) \ll \lambda_0$.

1. Establish the intensity $I(\Delta)$ and show that its expression differs from the previous intensity by a factor $\gamma(\Delta)$ called degree of temporal coherence. Express this factor.
2. Define and calculate the interference fringes contrast $C(\Delta)$.
3. Show on a graph $I(\Delta)$ as a function of Δ with its characteristic parameters. What is the beating period Δ_0 ? Express this period as a function of λ_0 and $\delta\lambda$.
4. Experimentally, we measure a period $\Delta_0 = 0,58$ mm. Knowing that the sodium doublet is centered on $\lambda_0 = 589$ nm, calculate $\delta\lambda$. Conclude on the interest of such interferometric setup.

2.3.2 Source of Gaussian spectral profile

Now, we put at S a quasi-monochromatic source, characterized by a Gaussian spectral profile :

$$\frac{dI}{d\sigma} = g(\sigma) = \frac{I_0}{W\sqrt{\pi}} \exp\left(-\frac{(\sigma - \sigma_0)^2}{W^2}\right) \tag{1}$$

where $\sigma = \frac{1}{\lambda}$, and σ_0 and $W \ll \sigma_0$ are positive constants. In the following, we assume that g can be extended to the negative values of σ.

1. Express the intensity $I(\Delta)$ as a function of Δ and the problem parameters. We will introduce the function $\widetilde{g}(x)$, Fourier transform of $g(\sigma)$, defined as :
$$\widetilde{g}(x) = \int_{-\infty}^{\infty} g(\sigma) \exp(2i\pi\sigma x)\, d\sigma \tag{2}$$
2. The following result is given :
$$\int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{a^2}\right) \exp(2i\pi u x)\, du = a\sqrt{\pi} \exp\left(-\pi^2 a^2 x^2\right) \tag{3}$$
Calculate $I(\Delta)$ and give the expression of the temporal coherence degree $\gamma(\Delta)$.
3. Show on a graph the shape of $I(\Delta)$ as a function of Δ, with its characteristic parameters, knowing that $\delta\sigma \ll \sigma_0$.
4. We admit that the interference contrast is clear enough until Δ_C, path difference corresponding to half of the maximum contrast, ie while $|\Delta| \leq \Delta_C$. Express Δ_C as a function of W, then as a function of λ_0 and $\delta\lambda$.
5. Δ_C is called coherence length. Explain the physical meaning of Δ_C and comment the interference condition $|\Delta| \leq \Delta_C$.
6. The Michelson interferometer is illuminated by a low pressure lamp of wavelength $\lambda_0 = 600$ nm and line width $\delta\lambda = 10^{-2}$ nm. Evaluate its coherence length. Answer to the same question for a laser of wavelength $\lambda_0 = 600$ nm and line width $\delta\lambda = 10^{-6}$ nm. Comment.

2.4 Spectral analysis of interferograms

The interferogram spectral analysis relies on the calculation of the Fourier transform $\mathcal{I}(\omega)$ of intensity $I(\Delta)$, defined by :

$$\mathcal{I}(\omega) = \int_0^{\Delta_{\max}} I(\Delta) \cos\left(\frac{\omega\Delta}{c}\right) d\Delta$$

1. Calculate $\mathcal{I}(\omega)$ in case of an ideal monochromatic source of angular velocity ω_0.
2. Show on a graph the shape of $\mathcal{I}(\omega)$ specifying :
   – the amplitudes as a function of I_0 and $\Delta_{\max}$;
   – the base widths $\delta\omega_{base}$ (see Figure 3) as a function of $\Delta_{\max}$ and c.

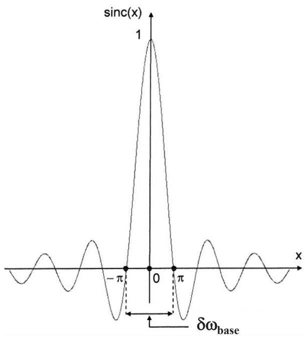

FIGURE 3 – Base width for function sinc(x).

3. What happens to $\mathcal{I}(\omega)$ when $\Delta_{\max}$ becomes very large ?
4. Without further calculations, deduce the shape of $\mathcal{I}(\omega)$ for a source emitting two waves of angular velocities ω_1 and ω_2, close to the average angular velocity ω_0, and of same intensity ($\omega_2 > \omega_1$).

In spectroscopy, a spectrometer is used to distinguish two radiation lines of angular velocities ω_1 and ω_2 close to each other, which depends on the resolution of the system. The common criterion to evaluate the order of magnitude of a spectrometer resolution is the Rayleigh criterion : the smallest measurable difference is obtained when two peaks of two different radiations are separated by half of the base width, ie when the main maximum of one peak corresponds to the first cancelation of the other.

5. Estimate the smallest spectral difference $\Delta\omega_R = \omega_2 - \omega_1$ observable by the system as a function of c and $\Delta_{\max}$.
6. The spectrometer resolution $\mathcal{R}$ is defined by $\mathcal{R} = \frac{\omega_0}{\Delta\omega_R}$. Show that $\mathcal{R}$ is fixed by the number of intensity maxima $N_{\max}$ recorded by the detector during the finite course of the interferometer. Comment.

3 Double Michelson interferometer : the wavelength meter

A wavelength meter (see Figure 4) allows a direct measurement of the wavelength of a stabilized laser. It appears as a double Michelson interferometer which compares the wavelength of an unknown stabilized laser with the known wavelength of a reference laser. The wavelength meter requires only one beam splitter (S_P), two identical corner cube reflectors and one mirror (M). All the reflection angles are equal to $\pi/4$.

"Corner cubes" are reflectors which have the property of reflecting light in the same direction as the incident light. They are made of glass of index $n = 1.5$ and each of the three opposite angles are equal to 90° with a better than arcsecond accuracy. A light beam hitting one of the three faces of the cube is reflected successively three times and therefore shifts slightly in position to stand alongside its incident direction.

The corner cube 2 is mobile and moves in a vacuum chamber. It is heavy enough to make negligible friction during translation. It is attached to a pulley of a stepper motor through a wire and is guided in a stainless steel tube. The designers have tried to get as close as the free fall.

The coherence lengths of the reference laser (Helium-Neon laser emitting at 632.8 nm) and the stabilized laser of unknown wavelength (CO_2 laser) are of the order of 300 m and 30 km, respectively.

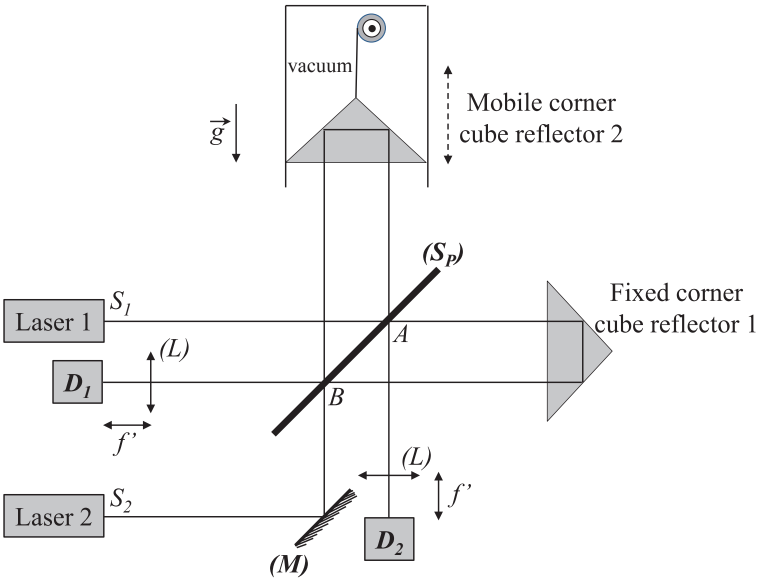

FIGURE 4 – Scheme of the wavelength meter principle.

1. The beam coming from Laser 1 hits the beam splitter at A and the central circular fringe of the interference pattern is detected by photodiode (D_1). Likewise, the beam coming from Laser 2 hits the beam splitter at B and the central circular fringe of the interference pattern is detected by photodiode (D_2). Compare the path differences for Laser 1 and Laser 2 at the two interference pattern centers (D_1) and (D_2), respectively.
2. Laser 1 is the reference laser of wavelength $\lambda_1 = 632.8$ nm. Laser 2 is a stabilized CO_2 laser of wavelength λ_2 to be measured. During the fall of corner cube 2, a counter connected to the two photodiodes evaluates to $p_1 = 3160556$ the number of sparklings detected by (D_1), and meanwhile, $p_2 = 188679$ sparklings are detected by (D_2). Evaluate λ_2.
3. Calculate the fall hight e of the mobile corner cube. Compare to the lasers coherence lengths and comment.
4. Determine the duration of the supposed free fall of the corner cube. Comment.
5. The fringes count is done with a one fringe accuracy. Considering that the reference wavelength λ_1 is known without uncertainty, give the relative uncertainty on the evaluation of λ_2. Comment.