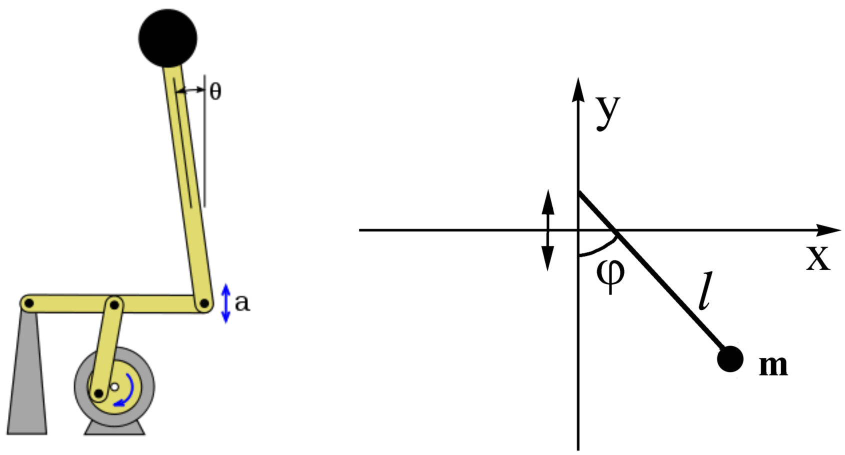
A funny oscillator

In this exercise, we study a mathematical pendulum, in which the pivot point vibrates in a fast vertical direction. The device can for example be realized by the setup plotted below (left panel). The right panel gives a schematic view of the system.



We denote

- ν the frequency of the vertical oscillations of the suspension
- a the amplitude of the oscillations of the suspension
- $\omega_0 = \sqrt{\frac{g}{l}}$
- g the free fall acceleration
- l the length of the pendulum
- m its mass
- ϕ the angle between the pendulum and the downwards direction
- The motion takes place in a plane, and we denote x the horizontal and y the vertical coordinate of the mass

1. Show that the y-coordinate of the mass is given by $y = -lcos\phi - acos\nu t$. What's the x-coordinate?
2. Calculate the potential energy V of the mass.
3. Calculate the kinetic energy T of the mass.
4. Give upper and lower bounds ($E_{pot}^{max}, E_{pot}^{min}$) for the potential energy.
5. Is the kinetic energy also bound?
6. Discuss the existence (or non-existence) of conserved quantities in the present system.
7. We indicate the following fact of Lagrangian mechanics: The Lagrangian function is given by

$$L(\phi, \dot{\phi}) = T - V \tag{1}$$

   The following differential equation is then equivalent to the equation of motion of the problem:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \tag{2}$$

   Show that the Lagrangian function for the present problem has the form

$$L = \frac{m}{2} l \dot{\phi}^2 + ml(g + a\nu^2 cos(\nu t))cos\phi + \frac{d}{dt} g(t) \tag{3}$$

   Calculate the function $g(t)$.

8. Show that the function g is irrelevant for the equation of motion of the present problem.
9. Derive the equation of motion for the present problem.
10. In the limit $a = 0$, interpret your findings.
11. Still in this limit, describe the trajectory of the pendulum if at time $t = 0$ the pendulum is given an energy $E > mgl$.

12. We now study the case $a << l$, $\nu >> \omega_0$. We aim at a perturbative solution for $\frac{a}{l}, \frac{\omega_0}{\nu} << 1$, at $\frac{a}{l}\frac{\nu}{\omega_0}$ fixed. Define

$$\delta = \frac{a}{l} sin\phi_0 cos(\nu t) \tag{4}$$

and write ϕ as a superposition of a slow and fast oscillation $\phi = \phi_0 + \delta$. Expand the equation of motion for ϕ_0 to first order in δ.

13. Calculate the time average of $\ddot{\phi}_0$ over a period of the fast motion.

14. Show that the time-averaged slow motion is given by

$$ml^2\ddot{\phi}_0 = -\frac{\partial}{\partial \phi_0} F(\phi_0) \tag{5}$$

with a function $F(\phi_0)$. Calculate this function.
Hint: If you do not succeed to derive the function F, use the following form for the remaining questions:

$$F(\phi_0) = const\left[-cos\phi_0 + \frac{a^2\nu^2}{4gl} sin\phi_0\right] \tag{6}$$

You can consider that the constant is positive.

15. What's the dimension of F? Give a physical interpretation of F.

16. Calculate the equilibrium position(s) of the mass.

17. Discuss their stability as a function of the various input parameters of the problem.

18. Give a physical interpretation of your findings.