Problem

1) It is well-known that the Heisenberg's uncertainty principle bounds from below the uncertainty in the measurement of the position and momentum for a particle. Consider the analog statement for the spin degrees of freedom. More precisely, consider a particle in the representation of $SU(2)$ of spin s, and define the uncertainty as $\Delta = \sum_{i=x,y,z} \Delta J_i^2 = \sum_i (\langle J_i^2 \rangle - \langle J_i \rangle^2)$. Show that

$$\hbar^2 s \leq \Delta \leq \hbar^2 s(s+1) \tag{1}$$

2) Consider the spin-coherent states

$$|\xi\rangle = e^{\frac{1}{\hbar}(\xi \hat{J}_- - \xi^* \hat{J}_+)} |s\rangle \tag{2}$$

where $\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$, $|s\rangle \equiv |J = s, J_z = s\rangle$, and ξ is a complex number, that we can write as $\xi = \frac{\theta}{2} e^{i\phi}$. Show that

$$\langle \xi | \hat{J}_i | \xi \rangle = \hbar s n_i \tag{3}$$

where n_i is the vector $(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

3) Show that the coherent states have the minimal possible uncertainty, and that $|\xi\rangle$ is an eigenvector of $\vec{n} \cdot \vec{J}$.

4) Using the alternative representation of the coherent state

$$|\xi\rangle = \frac{1}{(1 + |\mu|^2)^s} e^{\frac{1}{\hbar}\mu \hat{J}_-} |s\rangle \tag{4}$$

where $\mu = \tan(\frac{\theta}{2}) e^{i\phi}$, the overlap of two coherent states is

$$\langle \xi | \xi' \rangle = \left(\frac{1 + n \cdot n'}{2} \right)^s e^{isA(\hat{z}, n, n')} \tag{5}$$

where $A(a, b, c)$ is the area of the spherical triangle with vertices a, b, c. Show the absolute value part of this equality.
(Hint: start by finding out what is $\hat{J}_+ e^{\beta \hat{J}_-} |s\rangle$).

5) Consider the particle subject to a Hamiltonian

$$H = -B\mathbf{n}(\mathbf{t}) \cdot \mathbf{J} \tag{6}$$

where $\mathbf{n}(t)$ is a unit vector *slowly* varying with time, such that $\mathbf{n}(0) = \mathbf{n}(t_1) = \hat{\mathbf{z}}$. The particle starts in the state $|\psi(t=0)\rangle = |s\rangle$. Compute the total phase of the evolution, namely the overlap

$$\langle \psi(t=0) | \psi(t=t_1) \rangle$$

Problem.

We consider the quantum mechanics of a particle hopping on a one dimensional lattice with lattice spacing a. We denote the set of lattice sites as $\mathbf{L} = \{n\, a : n \in \mathbb{Z}\}$. The Hamiltonian is the following:

$$H = \frac{\hbar\omega}{2} \sum_{x \in \mathbf{L}} \left(|x\rangle\langle x+a| + |x+a\rangle\langle x| \right) , \tag{1}$$

where $\omega > 0$ is the hopping rate and $|x\rangle, x \in \mathbf{L}$ are the position eigenstates. They form an orthonormal basis of the Hilbert space: $\langle y|x\rangle = 1$ if $x = y$, and 0 otherwise.
Suppose that at $t = 0$, we measure the particle's position and find $x = 0$ as the outcome. Then we let particle evolve freely under (1) until $t > 0$. Then we measure the particle's position again, and obtain a random outcome x_t.
We repeat this experiment many times to obtain the probability distribution of x_t. As $t \to \infty$, which of the following statements are true?

1. The variance of the distribution has the asymptotic behavior $\mathrm{Var}(x) \sim t$.
2. The variance of the distribution has the asymptotic behavior $\mathrm{Var}(x) \sim t^2$.
3. The limit of the probability distribution $P(x_t/t)$ has a maximum at $x_t/t = 0$.
4. The limit of the probability distribution $P(x_t/t)$ has two maxima at $x_t/t = \pm v$ for some $v > 0$.
5. The limit of the probability distribution $P(x_t/t)$ is uniform in an interval $[-v, v]$ for some $v > 0$.

You can justify your answer by physical arguments and/or simple calculations.

Problem

Consider the quantum mechanics of a particle of mass m in a one-dimensional potential

$$V(x) = \begin{cases} 0 & |x| \leq a \\ U & |x| > a \end{cases} \tag{1}$$

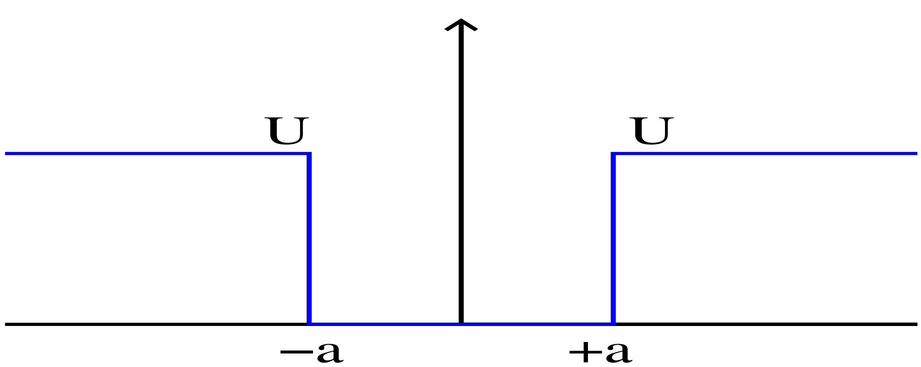
1. Suppose the particle is in its ground state and its energy is measured to be E_0. What is the potential amplitude U?

2. Find the smalest value of $U = U_1^*$ such that there is a second bounded eigenstate.

3. Find the smalest value of $U = U_n^*$ such that n bounded (traped) eigenstates exist

4. Supose that for $t < 0$, $U = U_1^* + \Delta U$ (with $U_1^* > \Delta U > 0$) and the particle is at the first excited state. At $t = 0$ the potential is changed to $U = U_1^* - \Delta U$. What is the probability that the particle will escape?

5. Supose now that for $t < 0$, $U = U_2^* + \Delta U$ (with $U_2^* > \Delta U > 0$) and the particle is at the second excited state. At $t = 0$ the potential is changed to $U = U_1^* - \Delta U$. What is the probability that the particle will escape?



Problem: One-dimensional traffic model

We consider a system of point-like cars moving along a line. At the initial time, the cars are uniformly and randomly distributed on the line with density 1. At time $t = 0$, each car has a random velocity. When a fast car catches up with a slow car, it starts moving at the velocity of the slow car; the two cars then form a group that moves together at the slow velocity. Since the cars are point-like, the group takes up no more space than a single car; it is as if the fast car had disappeared.

The goal of the exercise is to evaluate in different situations how the proportion of cars with a given velocity evolves over time.

Case with two velocities Assume that there are only two possible velocities, v_A and v_B with $v_A > v_B$. At the initial time, a fraction $\rho_0(A)$ of cars has velocity A and a fraction $\rho_0(B) = 1 - \rho_0(A)$ has velocity B.

1. Consider a car of type A given at the initial time. What is the probability that the first car of type B ahead of it is at a distance between x and $x + \mathrm{d}x$ away?

2. Determine the densities $\rho_t(A)$ and $\rho_t(B)$ of cars (or point-like groups of cars) of given type at any given time t.

Case with three velocities We assume now that there are three possible velocities $v_A > v_B > v_C$ with initial probabilities $\rho_0(A)$, $\rho_0(B)$ et $\rho_0(C)$.

3. Calculate $\rho_t(A)$, $\rho_t(B)$ and $\rho_t(C)$.

Generic case We assume now that each car has an initial velocity between v and $v + \mathrm{d}v$ with a probability $\rho_0(v)\mathrm{d}v$, for a given distribution $\rho_0(v)$.

4. Calculate $\rho_t(v)$, the density of cars (or groups of cars) at time t with a velocity v. (Note that this is not a distribution if $t \neq 0$...)

Boltzmann equation

5. Considering what happens between t and $t + \mathrm{d}t$, and assuming independence (just like in the Boltzmann equation), give an approximate relation for $\partial_t \rho_t(v)$.

6. Compare this last result to the exact answer.

Scaling law We assume that all velocities are positive ($\rho_0(v) = 0$ for $v < 0$) and that for small v one has $\rho_0(v) \simeq A v^\mu$.

7. Determine the behavior of $\rho_t(v)$ for $v \ll 1$ and large t.

8. Obtain an expression for the total density of groups of cars as a function of time at large times.